

## The drag-out problem in film coating theory

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### SUMMARY

An important step towards the understanding of many industrial coating processes is a solution of the drag-out problem, which is to determine the thickness of the film of liquid which clings to a plate when it is drawn steadily out of a bath of the liquid. An approximate solution, valid for small capillary numbers, was given by Landau and Levich, and considerable effort has been exerted to extend or refine this work. In this paper we show that the Landau-Levich result is an asymptotic solution valid as the capillary number tends to zero, a fact not properly appreciated hitherto, and show how correction terms may be obtained by the method of matched expansions. We also show how the results may be applied to the coating of a horizontal roller.

### 1. Introduction

Suppose an infinite flat plate is withdrawn vertically, with constant speed  $U$ , from a bath of liquid of viscosity  $\mu$  which has a horizontal free surface, and a steady state is established. What is the thickness of the film of liquid adhering to the plate at a large height above the free surface? This is the drag-out problem, and it is fundamental to the understanding of many industrial coating processes. For example, the coating of a horizontal roller, partially immersed in the liquid, which has been discussed by Middleman [1] and Tharmalingam and Wilkinson [2, 3], is only slightly more complicated. It will be shown that, within the parameter regime considered in this paper, the roller-coating problem reduces, in all but trivial respects, to the drag-out problem for a plate inclined at an appropriate angle  $\alpha$  to the vertical. The inclined plate problem is sketched in Figure 1.

Consider the fully developed region on a vertical plate. If we make the obvious force balance between gravity and viscosity, we find that the film thickness should be of order  $d$  where

$$d = (\mu U / \rho g)^{1/2}. \quad (1)$$

However, Landau and Levich [4] showed that when the capillary number  $Ca = \mu U / \gamma$  (where  $\gamma$  is the surface tension of the liquid) is small the thickness is actually

$$\bar{h} = \lambda d (Ca)^{1/6}. \quad (2)$$

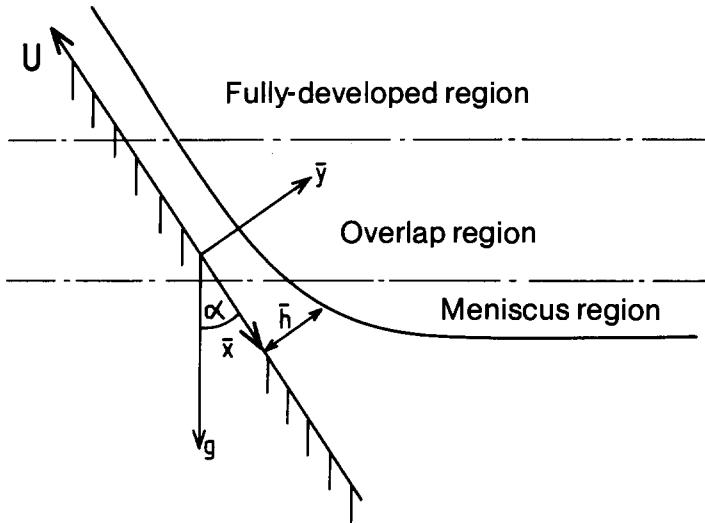


Figure 1. Plane drag-out at an angle  $\alpha$ .

Here  $\lambda$  is a dimensionless constant obtained from the numerical solution of a canonical ordinary differential equation describing the shape of the free surface in the overlap region. To obtain their result Landau and Levich assumed that the thickness was small in some sense, and the necessary condition  $Ca \ll 1$  emerged a posteriori. Since then various authors have attempted to improve the accuracy of the result (2) by retaining some of the previously neglected terms and by introducing new small terms. We may cite the work of White and Tallmadge [5], Spiers et al. [6] and Tharmalingam and Wilkinson [2] in this connection. All these authors noted that, that, with the retention of the extra terms in the equations, it becomes impossible to make the differential equation for the free surface satisfy appropriate conditions as the meniscus region is approached. To get round this various ad hoc alterations to the equations were made and, although in some cases the numerical agreement between theory and experiment was improved, the approach is basically irrational and does not clear up the difficulty.

It will be shown in this paper that the root of the problem lies in the ad hoc treatment of small terms by previous authors and, more particularly, in the neglect of certain small terms in the formula used in the overlap region for the free-surface curvature. If proper account is taken of the small terms, it is possible to obtain approximate solutions in the case  $Ca \ll 1$  by a rational, systematic procedure, namely the method of matched expansions. This puts the Landau-Levich theory on a firm foundation. Then the next approximation to (2), which involves a correction of order  $(Ca)^{1/2}$ , is calculated. As noted above, it is then a simple matter to obtain the solution to the roller-coating problem to the same order.

## 2. The overlap region

The configuration is sketched in Figure 1. The equations of motion, which are those of lubrication theory, are

$$0 = g \cos \alpha - \frac{1}{\rho} \frac{\partial p}{\partial \bar{x}} + \nu \frac{\partial^2 u}{\partial \bar{y}^2}, \quad (3)$$

$$0 = -g \sin \alpha - \frac{1}{\rho} \frac{\partial p}{\partial \bar{y}}. \quad (4)$$

The boundary conditions are

$$u = -U \quad \text{on} \quad \bar{y} = 0; \quad (5)$$

$$\frac{\partial u}{\partial \bar{y}} = 0, \quad p = -\gamma \kappa \quad \text{on} \quad \bar{y} = \bar{h}(\bar{x}). \quad (6)$$

Here  $\kappa$  is the curvature of the free surface  $\bar{y} = \bar{h}$  and is given by

$$\kappa = \frac{d^2 \bar{h}}{d\bar{x}^2} \left/ \left( 1 + \left( \frac{d\bar{h}}{d\bar{x}} \right)^2 \right)^{3/2} \right. . \quad (7)$$

These equations can be solved in a straightforward manner to give an ordinary differential equation for  $\bar{h}(\bar{x})$ . We integrate (4) and use (6) to find the pressure, substitute into (3) and integrate three times with respect to  $\bar{y}$ . This introduces the flux  $Q$ , which we write as

$$Q = -U\bar{H} + \frac{1}{3} \frac{\rho g \cos \alpha}{\mu} \bar{H}^3, \quad (8)$$

where  $\bar{H}$  is the film thickness at infinity up the slope, that is, as  $\bar{x} \rightarrow -\infty$ . (This is unknown at present of course). The equation for  $\bar{h}$  is

$$\frac{1}{3} \frac{\gamma}{\rho} \bar{h}^3 \frac{d\kappa}{d\bar{x}} + \frac{1}{3} g \cos \alpha (\bar{h}^3 - \bar{H}^3) - \frac{\mu U}{\rho} (\bar{h} - \bar{H}) - \frac{1}{3} g \sin \alpha \bar{h}^3 \frac{d\bar{h}}{d\bar{x}} = 0, \quad (9)$$

and we now consider the choice of dimensionless variables. We begin by scaling  $\bar{h}$  and  $\bar{H}$  on  $d$ , although this will turn out later to be inappropriate as we see from (2). There is no geometrical length scale for  $\bar{x}$ , so we choose that length which produces a balance between the curvature term  $d\kappa/d\bar{x}$  and the other important terms, namely those in  $U$  and  $g \cos \alpha$ . This length scale,  $l$ , say, turns out to be

$$l = d^{1/3} D^{2/3}, \quad (10)$$

where  $D = (\gamma/\rho g)^{1/2}$  is the length scale appropriate to a static meniscus and will therefore enter when we consider the meniscus region later. Note that

$$d/D = (Ca)^{1/2}. \quad (11)$$

We introduce a convenient small parameter  $\delta$  given by

$$\delta = (Ca)^{1/6} = (d/D)^{1/3} \quad (12)$$

and thus obtain the dimensionless form of (9),

$$h^3 \frac{d}{dx} \left\{ \frac{h_{xx}}{(1 + \delta^4 h_x^2)^{3/2}} \right\} = 3(h - H) - \cos \alpha (h^3 - H^3) + \delta^2 \sin \alpha h^3 h_x. \quad (13)$$

A number of small terms have been left out of the foregoing derivation, even though they are comparable with the small terms we have retained. These include the normal component of viscous stress at the free surface which would appear in (6) and was calculated by Spiers et al. [6]. This would produce a term of the order  $\delta^4$  in (13). However, it turns out that they are *uniformly* negligible if  $\delta \ll 1$ , at least to the order to which we shall work, whereas the small terms in (13) are not, and to keep the exposition as simple as possible we have displayed only those terms which are going to be important somewhere.

The equation (13) is third order and so four conditions are necessary to fix the solution and the unknown parameter  $H$ . We have

$$h \rightarrow H \quad \text{as} \quad x \rightarrow -\infty$$

and by linearising about  $H$  and, using the fact that  $\delta$  is small (and so is  $H$ , as will be shown), it can easily be shown that two exponentially growing solutions must be ruled out, giving two conditions. Next we note that (13) is invariant under a shift of origin of  $x$ , and this can be used to fix another constant, multiplying the exponentially decaying solution, equal to (say) unity. The solution must also merge somehow into a solution valid in the meniscus region. This matching, which yields the fourth condition, is essentially the subject of this paper. Before explaining this we shall use (13) to give a brief account of the Landau-Levich solution and the subsequent difficulty concerning the small terms which was mentioned above.

Suppose we put  $\alpha = 0$ , neglect the terms in  $\delta$ , and also suppose  $h$  is small. More precisely, we put  $h = H\tilde{h}$ ,  $x = 3^{-1/3} H\tilde{x}$  and suppose  $H$  is small. Then we obtain

$$\tilde{h}^3 \tilde{h}''' = \tilde{h} - 1 \quad (15)$$

with the boundary condition

$$\tilde{h} \rightarrow 1 \quad \text{as} \quad \tilde{x} \rightarrow -\infty.$$

Now as  $\tilde{x} \rightarrow +\infty$  we shall have

$$\frac{d^2 \tilde{h}}{d\tilde{x}^2} \rightarrow a \quad (\text{a constant}) \quad (16)$$

and the constant  $a$  can be found numerically. The quadratic behaviour of  $\tilde{h}$  as  $\tilde{x} \rightarrow \infty$  indicates

that the meniscus solution must approach the flat plate with zero contact angle, and it is easy to show that the (dimensional) curvature of the meniscus must be  $2^{1/2}/D$  at the point of contact. Comparing this with (16) we find

$$H = \frac{3^{2/3}a}{2^{1/2}} \left( \frac{d}{D} \right)^{1/3} = \frac{3^{2/3}a}{2^{1/2}} (Ca)^{1/6} \quad (17)$$

which of course is the result (2).

Now suppose we consider (13) again but retain what appear to be the next most important terms after those kept in (15). We obtain

$$\tilde{h}^3 \tilde{h}''' = \tilde{h} - 1 - \frac{1}{3} H^2 (\tilde{h}^3 - 1). \quad (18)$$

Now it seems that  $\tilde{h}$  cannot tend to infinity at all as  $\tilde{x} \rightarrow \infty$  because no matter how small  $H$  is the term in  $H^2$  on the right would ultimately dominate  $\tilde{h}$  and cause the third derivative  $\tilde{h}'''$  to change sign.

However it turns out that, when  $\tilde{h}$  takes on values large enough to bring the term in  $H^2$  in (18) to order one values, all the other small terms in (13) will also enter and a solution with acceptable behaviour can be found, and we now consider this in detail.

### 3. Solution by matched expansions

In view of (17) it will be necessary to scale  $h$ ,  $H$  and  $x$  by factors of  $\delta$ , and so we put

$$h = \delta\phi, \quad H = \delta J, \quad x = \delta\xi \quad (19)$$

and obtain

$$\phi^3 \frac{d}{d\xi} \left( \frac{\phi_{\xi\xi}}{(1 + \delta^4 \phi_{\xi}^2)^{3/2}} \right) = 3(\phi - J) - \delta^2 \cos \alpha (\phi^3 - J^3) + \delta^4 \sin \alpha \phi^3 \phi_{\xi} \quad (20)$$

Substituting the expansions

$$\phi = \phi_0 + \delta^2 \phi_1 \dots, \quad J = J_0 + \delta^2 J_1 \dots \quad (21)$$

into (20) we find

$$\phi_0^3 \phi_0''' = 3(\phi_0 - J_0) \quad (22)$$

with

$$\phi_0 \rightarrow J_0 \quad \text{as} \quad \xi \rightarrow -\infty.$$

As  $\xi \rightarrow \infty$  we have the asymptotic form

$$\phi_0' \sim 2A_0\xi + B_0 + \dots \quad (23)$$

in which the constant  $A_0$  has a definite value, given  $J_0$ , but  $B_0$  can arrange to take any value by a shift of the origin of  $\xi$ . (Note that (13) is invariant under a shift of the origin of  $x$ .)

At next order we find

$$\phi_0^3 \phi_1''' + 3\phi_0^2 \phi_0''' \phi_1 = 3(\phi_1 - J_1) - \cos \alpha (\phi_0^3 - J_0^3) \quad (24)$$

with

$$\phi_1 \rightarrow J_1 \quad \text{as} \quad \xi \rightarrow -\infty$$

and the asymptotic form, as  $\xi \rightarrow \infty$ ,

$$\phi_1' \sim -\frac{1}{2} \cos \alpha \xi^2 + 2A_1\xi + B_1 \dots \quad (25)$$

The constants  $B_0, A_1, B_1$  will be determined by matching to an 'outer' solution valid in the meniscus region. To obtain the correct scalings for this region we note that  $\phi_0 \sim \xi^2$  as  $\xi \rightarrow \infty$  and then we find that the small terms in (20) come into balance with the transformation

$$X = \delta^2 \xi, \quad \Phi = \delta^4 \phi \quad (26)$$

which gives

$$\frac{d}{dX} \left\{ \frac{\Phi_{XX}}{(1 + \Phi_X^2)^{3/2}} \right\} = -\cos \alpha + \sin \alpha \Phi_X + O(\delta^6), \quad (27)$$

This is just the equation of a hydrostatic meniscus, and the scale for  $\Phi$  and  $X$  is in fact  $D$ . In order to match correctly to the inner solution this meniscus must meet the flat plate at zero contact angle and we may take the origin of  $X$  at the point of contact. In other words, (27) must satisfy the conditions

$$\Phi = 0, \quad \Phi_X = 0 \quad \text{at} \quad X = 0. \quad (28)$$

To complete the solution we need a third condition and this is the requirement that the surface become horizontal at infinity:

$$\Phi \sim X \cot \alpha - C_1 \quad \text{as} \quad X \rightarrow \infty \quad (29)$$

where  $C_1$  is a constant.

Now (27) can be integrated twice to give

$$\frac{\Phi_{XX}}{(1 + \Phi_X^2)^{3/2}} = -X \cos \alpha + \Phi \sin \alpha + C_2, \quad (30)$$

$$-\frac{(\Phi_X \cos \alpha + \sin \alpha)}{(1 + \Phi_X^2)^{1/2}} = \frac{1}{2} (\Phi \sin \alpha - X \cos \alpha)^2 + C_2 (\Phi \sin \alpha - X \cos \alpha) + C_3, \quad (31)$$

where  $C_2, C_3$  are constants. Applying the conditions (28) we find

$$C_3 = -\sin \alpha \quad (32)$$

and applying (29) to (30) and (31) we find

$$C_2 = C_1 \sin \alpha, \quad (33)$$

$$C_2 = (2 - 2 \sin \alpha)^{1/2}. \quad (34)$$

What we need for the matching is the form of  $\Phi$  near  $X = 0$ , and this is now easily found to be

$$\Phi \sim \frac{1}{2} (2 - 2 \sin \alpha)^{1/2} X^2 - \frac{1}{6} \cos \alpha X^3 + \dots \quad (35)$$

Note that we can exploit the free choice of origin of  $X$  to place it *exactly* at a distance  $C_2$  above the undisturbed free surface (that is, to all orders in  $\delta$ ) and this means that (35) is the *complete* expansion of  $\Phi$  until terms of order  $\delta^6$  (which were suppressed in (27)) enter.

This derivation has been given for the case  $\alpha > 0$  and the results hold when  $\alpha = 0$  or  $\alpha < 0$ , although there is an apparent difficulty in the argument. When  $\alpha < 0$  we must let  $\alpha$  carry its own sign in (29) and proceed to the limit  $X \rightarrow -\infty$ , so that  $\Phi \rightarrow +\infty$  as before. When  $\alpha = 0$  the simplest procedure is to put  $\alpha = 0$  in (30) and (31) and abandon (29). The condition we need is that the solution becomes infinite at a finite value of  $X$ , not as  $X \rightarrow \infty$ , and this is equivalent to requiring the slope to be infinite and the curvature to be zero simultaneously. The left-hand side of (30) vanishes when  $X = C_2$  (when  $\alpha = 0$ ) and putting this into (31) with  $\Phi_X = \infty$  gives  $C_2 = 2^{1/2}$  as required.

Now matching (23 to (35) gives

$$2A_0 = (2 - 2 \sin \alpha)^{1/2}, \quad B_0 = 0. \quad (36)$$

It is possible to reduce (22) to the universal form (15) by trivial changes of variable, and then numerical integration yields

$$J_0(\alpha) = J_0(0)/(1 - \sin \alpha)^{1/2}, \quad (37)$$

where

$$J_0(0) = 0.94581 \dots,$$

a result given by previous authors, including Landau and Levich, although not to this accuracy.

Continuing now with (25) we see that the term  $-\frac{1}{2} \cos \alpha \xi^2$  matches with the cubic term in (35), and that  $A_1 = 0$ , and this requirement is sufficient to fix the unknown parameter  $J_1$  in (24). In this equation  $\alpha$  appears explicitly, and also implicitly because  $J_0$  depends on  $\alpha$ . The condition (25) also contains  $\cos \alpha$  but this of course is really the same  $\cos \alpha$  that appears in (24). Because  $J_0(\alpha)$  is known explicitly from (37) it is possible to reduce the numerical work on (24) to a single integration for the case  $\alpha = 0$ . This is done by removing a factor  $\cos \alpha (1 - \sin \alpha)^{-3/2}$  from  $\phi_1$  and  $J_1$ , and we find

$$J_1(\alpha) = J_1(0) \cos \alpha / (1 - \sin \alpha)^{3/2}, \quad (38)$$

where

$$J_1(0) = -0.10685 \dots$$

The final result for the film thickness can be written

$$\bar{H} = \left( \frac{\mu U}{\rho g} \right)^{1/2} \frac{1}{(1 - \sin \alpha)^{1/2}} \left\{ 0.94581 (Ca)^{1/6} - \frac{0.10685 \cos \alpha}{1 - \sin \alpha} (Ca)^{1/2} \dots \right\}. \quad (39)$$

#### 4. Horizontal roller

The configuration is sketched in Figure 2. This problem differs somewhat from the plane case because of the appearance of a natural length scale, namely the radius of the cylinder  $R$ , and because the film does not approach a constant thickness above the overlap region. However, it is easy enough to obtain an equation equivalent to (13) and this is

$$\epsilon h^3 \left( \frac{d^3 h}{d\theta^3} + \frac{dh}{d\theta} \right) = 3(h - H) + \sin \beta H^3 - \sin \theta h^3 + \frac{d}{R} \left( \cos \theta h^3 \frac{dh}{d\theta} - \frac{3}{8} \sin \theta h^4 \right) \quad (40)$$

Here we have a natural small parameter  $\epsilon$  given by

$$\epsilon = \frac{\gamma d}{\rho g R^3} = \frac{D^2 d}{R^3}. \quad (41)$$



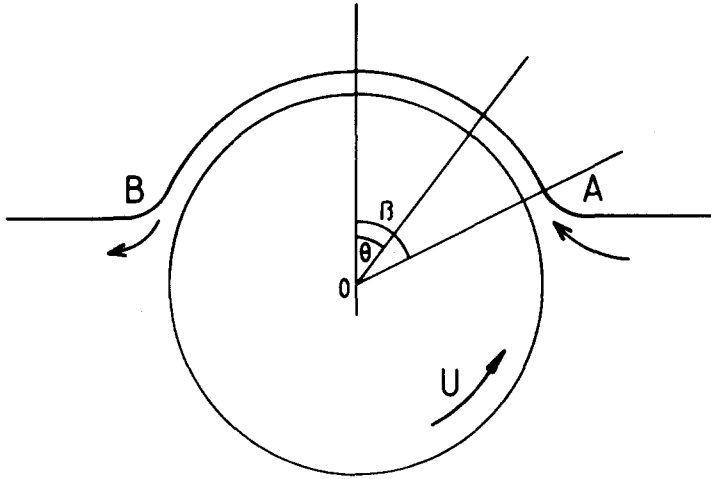


Figure 2. The horizontal roller in general position. The flow near  $A$  is analysed in the text. The flow near  $B$  is considerably more complicated.

One or two other remarks are in order here. We have put the flux  $Q$  in terms of the angle  $\beta$  by means of the expression

$$Q = -UH + \frac{1}{3} \frac{\rho g}{\mu} H^3 \sin \beta. \quad (42)$$

Then if  $\beta$  is the value of  $\theta$  at which the free surface joins the cylinder we have  $h = H$  at  $\theta = \beta$ , neglecting the small terms in  $\epsilon$  and  $d/R$  in (40), and so  $H$  is the (unknown) film thickness at the upper limit of the overlap region.

Also, the curvature  $\kappa$  has been approximated on the assumption that  $h \ll R$ , and we know from the previous section that this must fail as the meniscus is approached. There is, however, nothing to be gained by retaining the accurate expression for  $\kappa$  in polar coordinates, which is rather complicated, because these are not natural coordinates in the meniscus region. We shall return to this point later on.

In the overlap region we introduce scaled variables corresponding to (19):

$$h = \delta\phi, \quad H = \delta J, \quad \theta = \beta + \epsilon^{1/3} \delta \xi \quad (43)$$

and then (40) becomes

$$\begin{aligned} \phi^3 (\phi''' + \delta^2 \epsilon^{2/3} \phi') &= 3(\phi - J) + \delta^2 \{ \sin \beta J^3 - \sin (\beta + \epsilon^{1/3} \delta \epsilon) \phi^3 \} \\ &\quad - \delta^4 \left\{ \cos (\beta + \delta \epsilon^{1/3} \xi) \phi^3 \phi' - \frac{3}{8} \delta \epsilon^{1/3} \sin (\beta + \delta \epsilon^{1/3} \xi) \phi^4 \right\}, \end{aligned} \quad (44)$$

where we have used the result  $d/R\epsilon^{1/3} = \delta^2$ .

It is easy to see that the inner expansion (21) proceeds exactly as in the plane case, for the first two terms, with (of course)  $\beta = (\pi/2) - \alpha$ . When we introduce the scalings appropriate to the meniscus region, namely (26), it will be necessary, if the theory is to go through unchanged, that the argument of the sin and cos terms in (44), that is,  $\beta + \epsilon^{1/3} \delta \xi$ , does not change by order one. For this it is necessary that

$$\epsilon^{1/3} / \delta = D/R \ll 1 \quad (45)$$

and this is an extra assumption of the theory. This may be thought of as expressing the condition that the slope of the cylinder surface must not change significantly in the overlap region. Note that the appropriate independent variable in the meniscus region is not  $\delta^2 \xi$ , which would be a magnified polar angle, but a Cartesian coordinate in the direction of the local tangent.

The formula for the derivative of the curvature on the left of (44) has been obtained by suppressing a number of small terms, and it is necessary to check that none of these enter as far as order  $\delta^2$  – that is, in the 2-term inner expansion. This is straightforward but lengthy, and we shall not give the details here.

Now using the results of the previous section we find that at  $\theta = \beta$ ,

$$h = H = \delta J_0 + \delta^3 J_1 \dots \quad (46)$$

from (40), since the terms in  $\epsilon$  and  $d/R$  are negligible. Then the thickness at the top, which is what is usually measured, is given by

$$h(0) = H - \frac{1}{3} \sin \beta H^3,$$

or, in dimensional form, correct to  $O(\delta^3)$

$$\begin{aligned} \bar{H}(0) &= \left( \frac{\mu U}{\rho g} \right)^{1/2} \frac{1}{(1 - \sin \alpha)^{1/2}} \left\{ J_0 \delta + \frac{(J_1 - \frac{1}{3} J_0^3) \cos \alpha}{(1 - \sin \alpha)} \delta^3 \dots \right\} \\ &= \left( \frac{\mu U}{\rho g} \right)^{1/2} \frac{1}{(1 - \sin \alpha)^{1/2}} \left\{ 0.94581 (Ca)^{1/6} - \frac{0.38888 \cos \alpha}{(1 - \sin \alpha)} (Ca)^{1/2} \dots \right\}. \end{aligned} \quad (47)$$

## 5. Concluding remarks

Experimental work on the vertical drag-out of plates has been reported by White and Tallmadge [5] and Spiers et al. [6], and it is clear from these results that the Landau-Levich result is an overestimate of the film thickness. On the other hand the agreement is remarkably good considering that it is  $(Ca)^{1/6}$  that is supposed to be small; the discrepancy is not serious below  $Ca \approx 10^{-2}$ . This may have helped to conceal from these authors (and others) the fact that

the Landau-Levich result is really only asymptotically correct as  $Ca \rightarrow 0$ . In both papers just cited a comparison is made including points in the range  $1 < Ca < 10$ .

The correction in (39) is negative and this will tend to bring theory and experiment closer. How much closer is difficult to assess directly, since the raw experimental data are not available, the results being given on small scale graphs with one logarithmic axis. However White and Tallmadge give a 'corrected' formula which, although without any proper rational basis, has the same form as (39) but with a numerical coefficient 0.280 . . . instead of 0.10685. Their result gives a slight underestimate compared with the experimental points and so the present result is in marginally better (i.e. very good) agreement. Spiers et al. [6] solve their equation numerically and give a table comparing their results with the formula of White and Tallmadge [5]. The difference between the two is only a few per cent up to about  $Ca = 0.1$ , beyond which all the theories are meaningless anyway. We can conclude from this that the present result is in good agreement with experiment – that is, better than the Landau-Levich theory – in the appropriate range, but is not significantly better than the two earlier theories. It has the merit, which they do not have, of being the result of a correct treatment of the equations.

Data for the roller-coating problem have been presented by Middleman [1] and Tharmalingam and Wilkinson [2, 3]. Middleman's results show a downward trend in the film thickness as  $Ca$  increases which is in flat contradiction to all other reported results. The present author has been assured by Professor Middleman that his results have been correctly reported (that is, there is no error in the caption or anything of that kind) and at present there appears to be no explanation of the disagreement. The experimental results of Tharmalingam and Wilkinson show considerably more scatter than the flat plate results cited earlier. The present result, embodied in equation (47), falls between the curves marked 'Present Theory' and 'Theory 2' of reference [2], Figures 6 and 7, and gives good agreement with the experimental points. (N.B. The captions of these two figures are reversed.) There is similar agreement with the experimental data displayed in [3].

It may be appropriate to give a brief discussion of the non-uniformity in (39) as  $\alpha$  approaches  $\pi/2$ . This would be relevant to the coating of a deeply submerged roller. The results given are of course asymptotically valid as  $Ca \rightarrow 0$  with  $\alpha$  fixed.

The non-uniformity arises when the second term of (39) is comparable with the first, that is, when  $(\pi/2) - \alpha$  is of order  $\delta^2$ , so we begin by introducing a scaled angle  $S$  given by

$$\alpha = \frac{\pi}{2} - \delta^2 S. \quad (48)$$

The outer solution (35) now takes the form

$$\Phi \sim \frac{1}{2} \delta^2 S X^2 + \dots$$

and by comparing this with the solution of (13) at large  $x$  we can see that the appropriate scalings for (13) must be

$$h = \phi/\delta, \quad x = \xi/\delta, \quad H = J/\delta, \quad (49)$$

and these replace (19). Thus, the film thickness is larger by a factor  $1/\delta^2$  than when  $(\pi/2) - \alpha$  is order one, and moreover the scale for  $\xi$  turns out to be  $D$ , so that in fact  $\xi$  is identical with  $X$ . This means that the distinction between the inner and outer regions has disappeared, and a single equation describes the whole film, namely

$$\phi^3 \phi''' = 3(\phi - J) - S(\phi^3 - J^3) + \phi^3 \phi'. \quad (50)$$

A curious difficulty arises when we consider the boundary conditions. We need four to specify the solution and to fix  $J$ , which is unknown. As  $\xi \rightarrow +\infty$  we obtain from (50) the asymptotic solution

$$\phi \sim S\xi + A + B \exp(\xi) + C \exp(-\xi). \quad (51)$$

We must have  $B = 0$  and we can set  $A = 0$  arbitrarily (by choosing the origin of  $\xi$ ). The term  $S\xi$  embodies the requirement that the free surface become horizontal.

As  $\xi \rightarrow -\infty$  we must have  $\phi \rightarrow J$ , so putting  $\phi = J + \psi$  and linearising we find  $\psi$  is proportional to  $\exp(\alpha\xi)$ , where  $\alpha$  is any root of the cubic

$$\alpha^3 - \alpha - \frac{3}{J^3}(1 - SJ^2) = 0. \quad (52)$$

Those roots with negative real part must be excluded, so if there are two such roots the necessary four conditions will be forthcoming. Now it turns out that (52) always has one negative (real) root but only has two if  $SJ^2 < 1$ . Since  $J$  is unknown there is no guarantee of a unique solution. This is unwelcome, but there is no very obvious reason why the real, physical problem should have a unique solution; the film thickness might approach different steady values depending on the initial conditions.

A good deal of numerical work has been carried out and the results may be summarised as follows. As remarked above, there is (at most) one solution with  $SJ^2 < 1$ . This solution was located, and  $J$  calculated as a function of  $S$  but appears to exist only for  $S > 1.18$  approximately, at which point  $SJ^2$  was equal to 1. The region  $SJ^2 > 1$  was explored. Here, there is a free parameter in the solution at  $-\infty$ , and it appeared to be possible to achieve any value of  $J$  (for each  $S$ ) by suitable choice of this parameter. For complex  $\alpha$  the solutions oscillate violently before approaching the asymptote at  $+\infty$ , and it became very difficult to tell whether a given integration was behaving properly or not. The computations were abandoned on the grounds of expense.

The well-behaved solutions with  $SJ^2 < 1$  are the continuation of the asymptotic solutions described earlier, and one might speculate that *they* are not unique either. If not, the other solutions would have much larger values of the film thickness and would have a wavy character. Such flows do not seem to have been observed and might be unstable in some way.

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